Chapter 1

The Impossibility of a Satisfactory Population Ethics

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Population axiology concerns how to evaluate populations in regard to their goodness, that is, how to order populations by the relations "is better than" and "is as good as". This field has been riddled with paradoxes and impossibility results which seem to show that our considered beliefs are inconsistent in cases where the number of people and their welfare varies. All of these results have one thing in common, however. They all involve an adequacy condition that rules out Derek Parfit's *Repugnant Conclusion*. Moreover, some theorists have argued that we should accept the Repugnant Conclusion and hence that avoidance of this conclusion is not a convincing adequacy condition for a population axiology. As I shall show in this chapter, however, one can replace avoidance of the Repugnant Conclusion with a logically weaker and intuitively more convincing condition. The resulting theorem involves, to the best of my knowledge, logically weaker and intuitively more compelling conditions than the other theorems presented in the literature. As such, it challenges the very existence of a satisfactory population ethics.

1.1. Introduction

Population axiology concerns how to evaluate populations in regard to their goodness, that is, how to order populations by the relations "is better than" and "is as good as". This field has been riddled with impossibility results which seem to show that our considered beliefs are inconsistent in cases where the number of people and their welfare varies.[1] All of these results

[1] The informal Mere Addition Paradox in Parfit (1984), pp. 419ff is the *locus classicus*. For an informal proof of a similar result with stronger assumptions, see Ng (1989), p. 240. A formal proof with slightly stronger assumptions than Ng's can be found in Blackorby and Donaldson (1991). For theorems with much weaker assumptions, see my (1999), (2000b), and especially (2000a), (2001), and (2011).

have one thing in common, however. They all involve an adequacy condition that rules out Derek Parfit's Repugnant Conclusion:

> *The Repugnant Conclusion*: For any perfectly equal population with very high positive welfare, there is a population with very low positive welfare which is better, other things being equal.[2]

A few theorists have argued that we should accept the Repugnant Conclusion and hence that avoidance of this conclusion is not a convincing adequacy condition for a population axiology.[3] As I showed in Arrhenius (2003), however, one can replace avoidance of the Repugnant Conclusion in a version of Parfit's Mere Addition Paradox with a weaker condition, namely avoidance of the following conclusion:

> *The Very Repugnant Conclusion*: For any perfectly equal population with very high positive welfare, and for any number of lives with very negative welfare, there is a population consisting of the lives with negative welfare and lives with very low positive welfare which is better than the high welfare population, other things being equal.

This conclusion seems much harder to accept than the Repugnant Conclusions. Here we are comparing one population where everybody enjoys very high quality of lives with another population where people either have very low positive welfare or very negative welfare. Even if we were to accept the Repugnant Conclusion, we are not forced to accept the Very Repugnant Conclusion. We might, for example, accept the Repugnant Conclusion but not the Very Repugnant Conclusion because we give greater moral weight to suffering than to positive welfare.

In my 2003 paper, I made use of one controversial principle, namely a version of the Mere Addition Principle. I claimed that this principle could be replaced with other conditions that are intuitively much more compelling. To properly show this is the aim of this chapter. The theorem presented here involves, to the best of my knowledge, logically weaker and intuitively more compelling conditions than all the other impossibility

[2]See Parfit (1984), p. 388. My formulation is more general than Parfit's apart from that he does not demand that the people with very high welfare are equally well off. Expressions such as "a population with very high positive welfare", "a population with very low positive welfare", etc., are elliptical for the more cumbersome phrases "a population consisting only of lives with very high positive welfare", "a population consisting only of lives with very low positive welfare", etc.

[3]See e.g, Mackie (1985), Hare (1988), Tännsjö (1991, 1998, 2002), Ryberg (1996).

theorems presented in the literature.

In the present theorem we shall use a condition that is slightly logically stronger than avoidance of the Very Repugnant Conclusion but still intuitively very compelling:

> *The Weak Quality Addition Condition*: For any population X, there is a perfectly equal population with very high positive welfare, and a very negative welfare level, and a number of lives at this level, such that the addition of the high welfare population to X is at least as good as the addition of any population consisting of the lives with negative welfare and any number of lives with very low positive welfare to X, other things being equal.

Consider some arbitrary population X. Roughly, according to the above condition there is at least some number of people suffering horribly, and some number of people enjoying excellent lives, such that it is better to add the people with the excellent lives to X rather than the suffering lives and any number of lives barely worth living.

The Weak Quality Addition Condition implies avoidance of the Very Repugnant Conclusion. An example of a principle that violates this condition is Total Utilitarianism according to which a population is better than another if and only if it has greater total welfare. Consider the following populations:

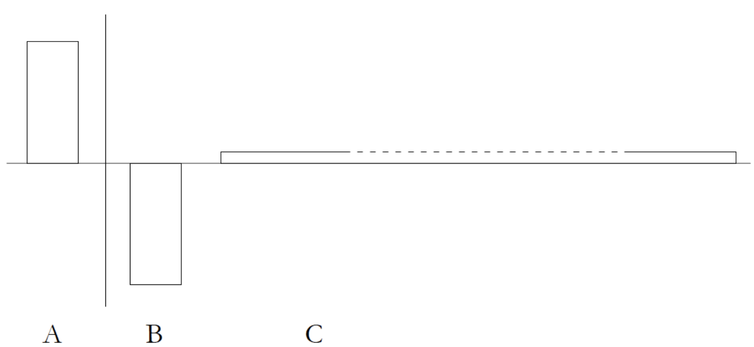

Diagram 1

The blocks in the above diagram represent three populations, A, B and C. The width of each block represents the number of people in the corresponding population, the height represents their welfare. Dashes indicate that the block in question should intuitively be much wider than shown, that is, the population size is intuitively much larger than shown (in this case population C).

The A-people have very high positive welfare, the B-people have very negative welfare, and the C-people have very low positive welfare. However, if there is just sufficiently many C-people, the total well-being in B∪C will be higher than in A. Thus, Total Utilitarianism ranks B∪C as better than A. This holds irrespective of how much people suffer in B and of how many they are.

1.2. The Basic Structure

For the purpose of proving the theorem, it will be useful to state some definitions and assumptions, and introduce some notational conventions. A *life* is individuated by the person whose life it is and the kind of life it is. A *population* is a finite set of lives in a possible world.[4] We shall assume that for any natural number n and any welfare level $\mathbf{X}$, there is a possible population of n people with welfare $\mathbf{X}$. Two populations are identical if and only if they consist of the same lives. Since the same person can exist (be instantiated) and lead the same kind of life in many different possible worlds, the same life can exist in many possible worlds. Moreover, since two populations are identical exactly if they consist of the same lives, the same population can exist in many possible worlds. A *population axiology* is an "at least as good as" quasi-ordering of all possible populations, that is, a reflexive, transitive, but not necessarily complete ordering of populations in regard to their goodness.

A, B, C,..., A_1, A_2,..., A_n, A∪B, and so on, denote populations of finite size. The number of lives in a population X (X's population size) is given by the function $N(\mathrm{X})$. We shall adopt the convention that populations represented by different letters, or the same letter but different indexes, are pairwise disjoint. For example, $\mathrm{A}\cap\mathrm{B} = \mathrm{A}_1\cap\mathrm{A}_2 = \emptyset$.

The relation "*has at least as high welfare as*" quasi-orders (being reflexive, transitive, but not necessarily complete) the set $\mathbf{L}$ of all possible lives. A life p_1 has higher welfare than another life p_2 if and only if p_1 has at least as high welfare as p_2 and it is not the case that p_2 has at least as high welfare as p_1. A life p_1 has the same welfare as p_2 if and only if p_1 has at

[4]For some possible constraints on possible populations, see Arrhenius (2000a; 2011, Chapter 2).

least as high welfare as p_2 and p_2 has at least as high welfare as p_1.

We shall also assume that there are possible lives with positive or negative welfare. We shall say that a life has *neutral welfare* if and only if it is equally good for the person living it as a neutral welfare component (that is, a component that neither makes her life better, nor worse for her), and that a life has *positive* (*negative*) welfare if and only if it has higher (lower) welfare than a life with neutral welfare.[5]

By a *welfare level* **A** we shall mean a set such that if a life a is in **A**, then a life b is in **A** if and only if b has the same welfare as a. In other words, a welfare level is an equivalence class on **L**. Let a^* be a life which is representative of the welfare level **A**. We shall say that a welfare level **A** is higher (lower, the same) than (as) a level **B** if and only if a^* has higher (lower, the same) welfare than (as) b^*; that a welfare level **A** is positive (negative, neutral) if and only if a^* has positive (negative, neutral) welfare; and that a life b has welfare above (below, at) **A** if and only if b has higher (lower, the same) welfare than (as) a^*.

We shall assume that *Discreteness* is true of the set of all possible lives **L** or some subset of **L**:

> *Discreteness*: For any pair of welfare levels **X** and **Y**, **X** higher than **Y**, the set consisting of all welfare levels **Z** such that **X** is higher than **Z**, and **Z** is higher than **Y**, has a finite number of members.

The statement of the informal version of some of the adequacy conditions below, for example the Non-Elitism Condition, involve the not so exact relation "slightly higher welfare than". In the exact statements of those adequacy conditions, we shall instead make use of two consecutive welfare levels, that is, two welfare levels such that there is no welfare level in between them. Discreteness ensures that there are such welfare levels. Intuitively speaking, if **A** and **B** are two consecutive welfare levels, **A** higher than **B**, then **A** is just slightly higher than **B**. More importantly, the in-

[5]A welfare component is neutral relative to a certain life x iff x with this component has the same welfare as x without this component. A hedonist, for example, would typically say that an experience which is neither pleasurable nor painful is neutral in value for a person and as such does not increase or decrease the person's welfare. The above definition can of course be combined with other welfarist axiologies, such as desire and objective list theories. For a discussion of alternative definitions of a neutral life, many of which would also work fine in the present context, see Arrhenius (2011), Chapter 2. Notice that we actually do not need an analysis of a neutral welfare in the present context but rather just a criterion, and the criterion can vary with different theories of welfare.

tuitive plausibility of the adequacy conditions is preserved. Of course, this presupposes that the order of welfare levels is fine-grained, which is exactly what is suggested by expressions such as "Marc is slightly better off than Vito" and the like. Notice that Discreteness does not exclude the view that for any welfare level, there is a higher and a lower welfare level (compare with the integers).

Discreteness can be contrasted with Denseness:

> *Denseness*: There is a welfare level in between any pair of distinct welfare levels.

My own inclination is that Discreteness rather than Denseness is true. If the latter is true, then for any two lives p_1 and p_2, p_1 with higher welfare than p_2, there is a life p_3 with welfare in between p_1 and p_2, and a life p_4 with welfare in between p_3 and p_2, and so on *ad infinitum*. It is improbable, I think, that there are such fine discrimination between the welfare of lives, even in principle. Rather, what we will find at the end of such a sequence of lives is a pair of lives in between which we cannot find any life or only lives with roughly the same welfare as both of them.

One might think otherwise, and a complete treatment of this topic would involve a detailed examination of the features of different welfarist axiologies. We shall not engage in such a discussion here. The important question is whether the validity and plausibility of the theorem below depend on whether Denseness or Discreteness is true. But that is not the case (indeed, it would have been an interesting result if the existence of a plausible axiology hinged on whether Denseness or Discreteness is true). If Denseness is true of the set of all possible lives $\mathbf{L}$, then we can form a subset $\mathbf{L}_1$ of $\mathbf{L}$ such that Discreteness is true of $\mathbf{L}_1$, and such that all the conditions which are intuitively plausible in regard to populations which are subsets of $\mathbf{L}$ are also intuitively plausible in regard to populations which are subsets of $\mathbf{L}_1$. Given that Denseness is true of $\mathbf{L}$, one cannot plausibly deny that there is such a subset $\mathbf{L}_1$ since the order of the welfare levels in $\mathbf{L}_1$ could be arbitrarily fine-grained even though Discreteness is true of $\mathbf{L}_1$. Now, since all the populations which are subsets of $\mathbf{L}_1$ also are subsets of $\mathbf{L}$, if we can show that there is no population axiology satisfying the adequacy conditions in regard to the populations which are subsets of $\mathbf{L}_1$, then it follows that there is no population axiology satisfying the adequacy conditions in regard to the populations which are subsets of $\mathbf{L}$.

Given Discreteness, we can index welfare levels with integers in a natural manner. Discreteness in conjunction with the existence of a neutral welfare level and a quasi-ordering of lives implies that there is at least one positive welfare level in $\mathbf{L}$ such that there is no lower positive welfare level.[6] Let $\mathbf{W}_1$, $\mathbf{W}_2$, $\mathbf{W}_3, \ldots$ and so forth represent positive welfare levels, starting with one of the positive welfare level for which there is no lower positive one, such that for any pair of welfare levels $\mathbf{W}_n$ and $\mathbf{W}_{n+1}$, $\mathbf{W}_{n+1}$ is higher than $\mathbf{W}_n$, and there is no welfare level $\mathbf{X}$ such that $\mathbf{W}_{n+1}$ is higher than $\mathbf{X}$, and $\mathbf{X}$ is higher than $\mathbf{W}_n$. Analogously, let $\mathbf{W}_{-1}$, $\mathbf{W}_{-2}$, $\mathbf{W}_{-3}, \ldots$ and so on represent negative welfare levels.[7] The neutral welfare level is represented by $\mathbf{W}_0$.

A *welfare range* $\mathbf{R}(x, y)$ is a union of at least *three* welfare levels defined by two welfare levels $\mathbf{W}_x$ and $\mathbf{W}_y$, $x < y$, such that for any welfare level $\mathbf{W}_z$, $\mathbf{W}_z$ is a subset of $\mathbf{R}(x, y)$ if and only if $x \leq z \leq y$.[8] We shall say that a welfare range $\mathbf{R}(x, y)$ is higher (lower) than another range $\mathbf{R}(z, w)$ if and only if $x > w$ ($y < z$); that a welfare range $\mathbf{R}(x, y)$ is positive (negative) if and only if $x > 0$ ($y < 0$); and that a life p has welfare above (below, in) $\mathbf{R}(x, y)$ if and only if p is in some $\mathbf{W}_z$ such that $z > y$ ($z < x$, $y \geq z \geq x$).

1.3. Adequacy Conditions

We shall make use of the following five adequacy conditions:

> *The Egalitarian Dominance Condition*: If population A is a perfectly equal population of the same size as population B, and every person in A has higher welfare than every person in B, then A is better than B, other things being equal.

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[6] There might be more than one since we only have a quasi-ordering of lives, that is, there might be lives and thus welfare levels which are incomparable in regard to welfare.

[7] Another way to put it is that we have a division of welfare levels into threads, each of which is completely ordered. If we assume that all lives with neutral welfare are comparable and have the same welfare level, which seems natural given our definition of a neutral welfare level, then the neutral welfare level is comparable with all other welfare levels, irrespective of the thread to which the latter level belongs. Moreover, any negative welfare level in any thread can be compared to any positive level in any thread. Both of these implications are desirable. For example, it would be odd to claim that p_1 enjoys positive welfare and p_2 suffers negative welfare but p_1 and p_2 are incomparable in regard to welfare. Still, two positive welfare levels might be incomparable, and two negative welfare levels might be incomparable. I am grateful to Kaj Børge Hansen for pressing this issue.

[8] The reason for restricting welfare ranges to unions of at least three welfare levels, as opposed to at least two welfare levels, is that this restriction allows us to simplify the exact statements of the adequacy conditions.

The Egalitarian Dominance Condition (exact formulation): For any populations A and B, N(A)=N(B), and any welfare level $\mathbf{W}_x$, *if* all members of B have welfare below $\mathbf{W}_x$, and A$\subset\mathbf{W}_x$, *then* A is better than B, other things being equal.

The General Non-Extreme Priority Condition: There is a number n of lives such that for any population X, and any welfare level **A**, a population consisting of the X-lives, n lives with very high welfare, and one life with welfare **A**, is at least as good as a population consisting of the X-lives, n lives with very low positive welfare, and one life with welfare slightly above **A**, other things being equal.

The General Non-Extreme Priority Condition (exact formulation): For any $\mathbf{W}_z$, there is a positive welfare level $\mathbf{W}_u$, and a positive welfare range $\mathbf{R}(1,\ y)$, $u > y$, and a number of lives $n > 0$ such that *if* A$\subset\mathbf{W}_x$, $x \geq u$, B$\subset\mathbf{R}(1, y)$, N(A)=N(B)=n, C$\subset\mathbf{W}_z$, D$\subset\mathbf{W}_{z+1}$, N(C)= N(D)=1, *then,* for any E, A$\cup$C$\cup$E is at least as good as B$\cup$D$\cup$E, other things being equal.

The Non-Elitism Condition: For any triplet of welfare levels **A**, **B**, and **C**, **A** slightly higher than **B**, and **B** higher than **C**, and for any one-life population A with welfare **A**, there is a population C with welfare **C**, and a population B of the same size as A$\cup$C and with welfare **B**, such that for any population X consisting of lives with welfare ranging from **C** to **A**, B$\cup$X is at least as good as A$\cup$C$\cup$X, other things being equal.

The Non-Elitism Condition (exact formulation): For any welfare levels $\mathbf{W}_x$, $\mathbf{W}_y$, $x - 1 > y$, there is a number of lives $n > 0$ such that *if* A$\subset\mathbf{W}_x$, N(A)=1, B$\subset\mathbf{W}_y$, N(B)=n, and C$\subset\mathbf{W}_{x-1}$, N(C)=$n+1$, *then*, for any D$\subset\mathbf{R}(y, x)$, C$\cup$D is at least as good as A$\cup$B$\cup$D, other things being equal.

The Weak Non-Sadism Condition: There is a negative welfare level and a number of lives at this level such that an addition of any number of people with positive welfare is at least as good as an addition of the lives with negative welfare, other things being equal.

The Weak Non-Sadism Condition (exact formulation): There is a welfare level $\mathbf{W}_x$, $x < 0$, and a number of lives n, such that *if* A$\subset\mathbf{W}_x$,

N(A)=n, B$\subset$$\mathbf{W}_y$, $y > 0$, *then*, for any population C, B$\cup$C is at least as good as A$\cup$C, other things being equal.

The Weak Quality Addition Condition: For any population X, there is a perfectly equal population with very high positive welfare, and a very negative welfare level, and a number of lives at this level, such that the addition of the high welfare population to X is at least as good as the addition of any population consisting of the lives with negative welfare and any number of lives with very low positive welfare to X, other things being equal.

The Weak Quality Addition Condition (exact formulation): For any population X, there is a negative welfare level $\mathbf{W}_x$, $x < 0$, two positive welfare ranges $\mathbf{R}(u, v)$ and $\mathbf{R}(1, y)$, $u > y$, and two population sizes $n > 0$, $m > 0$, such that *if* A$\subset$$\mathbf{W}_z$, $z \geq u$, N(A)=n, B$\subset$$\mathbf{R}(1, y)$, C$\subset$$\mathbf{W}_x$, N(C)=m *then* A$\cup$X is at least as good as B$\cup$C$\cup$X, other things being equal.

Notice that in the exact formulation of the adequacy conditions, we have eliminated concepts such as "very high positive welfare", "very low positive welfare", "very negative welfare", and the like. Hence, such concepts are not essential for our discussion and results. For example, in the exact formulation of the Weak Quality Addition Condition, we have eliminated the concepts "very low positive welfare" and "very high positive welfare" and replaced them with two non-fixed positive welfare ranges, one starting at the lowest positive welfare level, and the other one starting anywhere above the first range.

1.4. The Impossibility Theorem

The Impossibility Theorem: There is no population axiology which satisfies the Egalitarian Dominance, the General Non-Extreme Priority, the Non-Elitism, the Weak Non-Sadism, and the Weak Quality Addition Condition.

Proof. We shall show that the contrary assumption leads to a contradiction. We shall first prove two lemmas to the effect that the Non-Elitism and the General Non-Extreme Priority Conditions each imply another condition, Condition β and Condition δ respectively. We shall then prove a lemma to the effect that Weak Quality Addition Condition and Condition δ imply what we shall call the Restricted Quality Addition Condition. Finally, we shall then show that there is no population axiology which satisfies this condition in conjunction with Conditions β and δ, the Egalitarian Dominance, and the Weak Non-Sadism Condition.

1.4.1. *Lemma 1.1*

Lemma 1.1: The Non-Elitism Condition implies Condition β.

Condition β: For any triplet $\mathbf{W}_x$, $\mathbf{W}_y$, $\mathbf{W}_z$ of welfare levels, $x > y > z$, and any number of lives $n > 0$, there is a number of lives $m > n$ such that *if* A$\subset\mathbf{W}_x$, N(A)=n, B$\subset\mathbf{W}_z$, N(B)=m, and C$\subset\mathbf{W}_y$, N(C)=$m+n$, *then,* for any D$\subset\mathbf{R}(z, y+1)$, C$\cup$D is at least as good as A$\cup$B$\cup$D, other things being equal.

We shall prove Lemma 1.1 by first proving

Lemma 1.1.1: The Non-Elitism Condition entails Condition α.

Condition α: For any welfare levels $\mathbf{W}_x$, $\mathbf{W}_y$, $x-1 > y$, and for any number of lives $n > 0$, there is a number of lives $m \geq n$ such that *if* A$\subset\mathbf{W}_x$, N(A)=n, B$\subset\mathbf{W}_y$, N(B)=m, C$\subset\mathbf{W}_{x-1}$, N(C)=$m+n$, *then*, for any D$\subset\mathbf{R}(y, x)$, C$\cup$D is at least as good as A$\cup$B$\cup$D, other things being equal.

Proof: Let

(1) $\mathbf{W}_x$ and $\mathbf{W}_y$ be any welfare levels such that $x-1 > y$;
(2) n be any number of lives such that $n > 0$;
(3) $p > 0$ be a number which satisfies the Non-Elitism Condition for $\mathbf{W}_x$ and $\mathbf{W}_y$.

Let A$_1$,..., A$_{n+1}$, B$_1$,..., B$_{n+1}$, and C$_0$,..., C$_n$, be any three sequences of populations satisfying

(4) A$_i\subset\mathbf{W}_x$; N(A$_i$)=1 for all i, $1 \leq i \leq n$; A$_{n+1} = \emptyset$;
(5) B$_i\subset\mathbf{W}_y$; N(B$_i$) $= p$, for all i, $1 \leq i \leq n$; B$_{n+1} = \emptyset$;
(6) C$_i\subset\mathbf{W}_{x-1}$; N(C$_i$) $= p+1$, for all i, $1 \leq i \leq n$; C$_0 = \emptyset$.

Finally, let

(7) D be any population such that $D \subset \mathbf{R}(y, x)$.

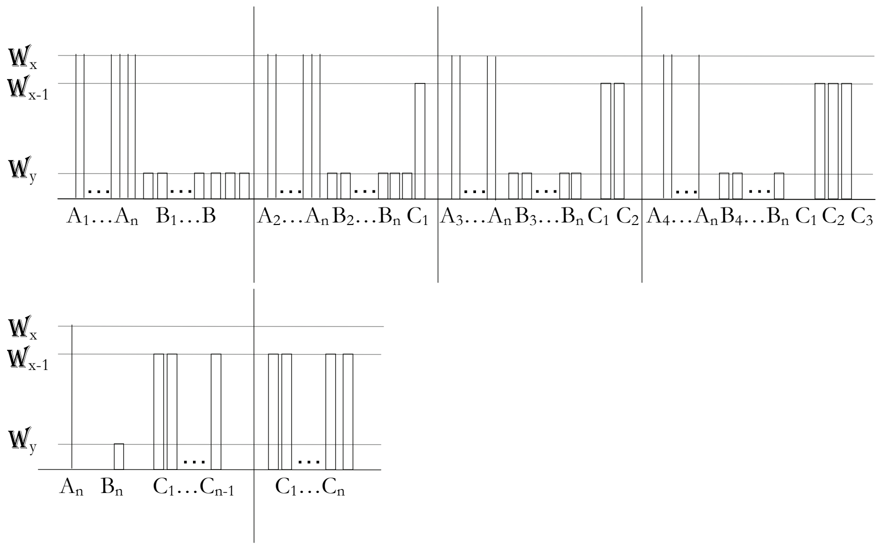


Diagram 2

The above diagram shows a selection of the involved populations in a case where $n \geq 6$. Dots in between two blocks indicate that there is a number of same sized blocks which have been omitted from the diagram. Population D is omitted throughout.

Recall that we have adopted the convention that populations represented by different letters, or the same letter but different indexes, are pairwise disjoint. Hence, for example, $A_1 \cap A_2 = \emptyset$.

Since $\mathbf{W}_x$ and $\mathbf{W}_y$ can be any pair of welfare levels separated by at least one welfare level, and D can be any population consisting of lives with welfare ranging from $\mathbf{W}_y$ to $\mathbf{W}_x$, and $N(A_1 \cup \ldots \cup A_n) = n$ (by (4)) can be any number of lives greater than zero, and $N(B_1 \cup \ldots \cup B_n) = np \geq n$ (by (5)), we can show that Lemma 1.1.1 is true by showing that $C_1 \cup \ldots \cup C_n \cup D$ is at least as good as $A_1 \cup \ldots \cup A_n \cup B_1 \cup \ldots \cup B_n \cup D$. This suffices since $A_1, \ldots, A_{n+1}$, $B_1, \ldots, B_{n+1}$, $C_1, \ldots, C_n$, and D are arbitrary populations satisfying (4)-(7).

It follows from (3)-(6) and the Non-Elitism Condition that

(8) $C_i \cup E$ is at least as good as $A_i \cup B_i \cup E$ for all i, $1 \leq i \leq n$ and any

E$\subset\mathbf{R}(y,x)$

and from (4)-(7) that

(9) $\mathrm{A}_{i+1}\cup\ldots\cup\mathrm{A}_{n+1}\cup\mathrm{B}_{i+1}\cup\ldots\cup\mathrm{B}_{n+1}\cup\mathrm{C}_0\cup\ldots\cup\mathrm{C}_{i-1}\cup\mathrm{D}\subset\mathbf{R}(y,x)$ for all i, $1\leq i\leq n$.

Letting $\mathrm{E}=\mathrm{A}_{i+1}\cup\ldots\cup\mathrm{A}_{n+1}\cup\mathrm{B}_{i+1}\cup\ldots\cup\mathrm{B}_{n+1}\cup\mathrm{C}_0\cup\ldots\cup\mathrm{C}_{i-1}\cup\mathrm{D}$, (8) and (9) imply that

(10) $\mathrm{C}_i\cup[\mathrm{A}_{i+1}\cup\ldots\cup\mathrm{A}_{n+1}\cup\mathrm{B}_{i+1}\cup\ldots\cup\mathrm{B}_{n+1}\cup\mathrm{C}_0\cup\ldots\cup\mathrm{C}_{i-1}\cup\mathrm{D}]$ is at least as good as $\mathrm{A}_i\cup\mathrm{B}_i\cup[\mathrm{A}_{i+1}\cup\ldots\cup\mathrm{A}_{n+1}\cup\mathrm{B}_{i+1}\cup\ldots\cup\mathrm{B}_{n+1}\cup\mathrm{C}_0\cup\ldots\cup\mathrm{C}_{i-1}\cup\mathrm{D}]$ for all i, $1\leq i\leq n$ (see Diagram 2).

Transitivity and (10) yield

(11) $\mathrm{C}_n\cup\mathrm{A}_{n+1}\cup\mathrm{B}_{n+1}\cup\mathrm{C}_0\cup\ldots\cup\mathrm{C}_{n-1}\cup\mathrm{D}$ is at least as good as $\mathrm{A}_1\cup\mathrm{B}_1\cup\mathrm{A}_2\cup\ldots\cup\mathrm{A}_{n+1}\cup\mathrm{B}_2\cup\ldots\cup\mathrm{B}_{n+1}\cup\mathrm{C}_0\cup\mathrm{D}$

and since $\mathrm{A}_{n+1}=\mathrm{B}_{n+1}=\mathrm{C}_0=\emptyset$ by (4)-(6), line (11) is equivalent to (see Diagram 2)

(12) $\mathrm{C}_1\cup\ldots\cup\mathrm{C}_n\cup\mathrm{D}$ is at least as good as $\mathrm{A}_1\cup\ldots\cup\mathrm{A}_n\cup\mathrm{B}_1\cup\ldots\cup\mathrm{B}_n\cup\mathrm{D}$. □

To show that Lemma 1.1 is true, we now need to prove

Lemma 1.1.2: Condition α entails Condition β.

Proof**.** Let

(1) $\mathbf{W}_x$, $\mathbf{W}_y$, $\mathbf{W}_z$ be any three welfare levels such that $x>y>z$;
(2) $r=x-y$.

Let $\mathrm{A}_1,\ldots,\mathrm{A}_{r+1}$ and $\mathrm{B}_1,\ldots,\mathrm{B}_{r+1}$ be any two sequences of populations, $m_0,\ldots,m_r$ any sequence of integers, and f a function satisfying

(3) $m_0>0$;
(4) $f(m_i)=m_0+m_1+\ldots+m_i$, for all i, $0\leq i\leq r$;
(5) $m_i\geq f(m_{i-1})$ satisfies Condition α for $\mathbf{W}_{x-(i-1)}$, $\mathbf{W}_z$, and $f(m_{i-1})$ for all i, $1\leq i\leq r$;

(6) $\mathrm{A}_i \subset \mathbf{W}_{x-(i-1)}$, $N(\mathrm{A}_i) = f(m_{i-1})$ for all i, $1 \leq i \leq r+1$;
(7) $\mathrm{B}_i \subset \mathbf{W}_z$, $N(\mathrm{B}_i) = m_i$, for all i, $1 \leq i \leq r$; $\mathrm{B}_{r+1} = \emptyset$.

Finally, let

(8) D be any population such that $\mathrm{D} \subset \mathbf{R}(z, y+1)$.

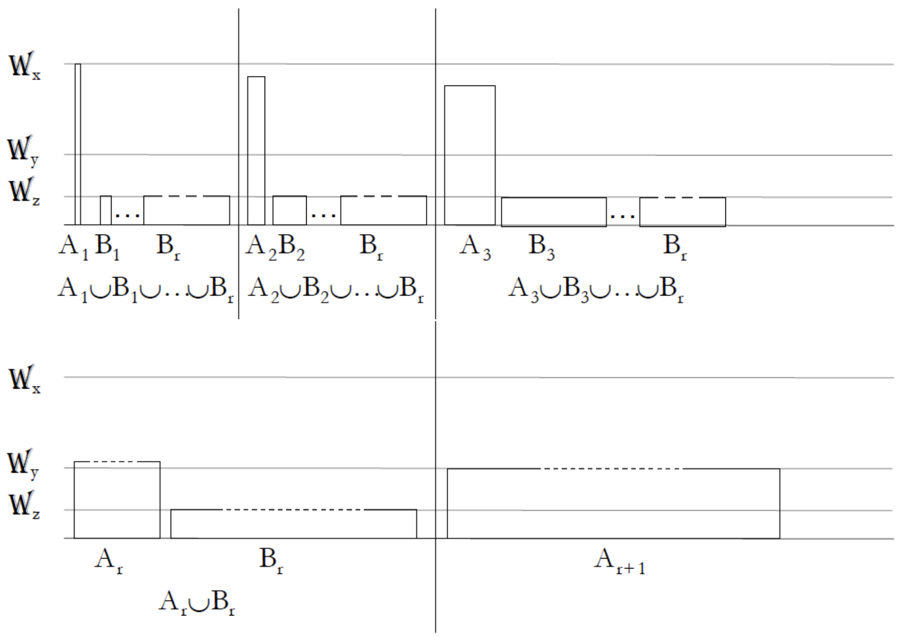


Diagram 3

The above diagram shows a selection of the involved populations in a case where $r \geq 4$. Population D is omitted throughout.

We can conclude from (3)-(7) that $N(\mathrm{B}_1 \cup \ldots \cup \mathrm{B}_r) > m_0 = N(\mathrm{A}_1)$. Consequently, since $\mathbf{W}_x$, $\mathbf{W}_y$, and $\mathbf{W}_z$ can be any welfare levels such that $x > y > z$, and D can be any population consisting of lives with welfare ranging from $\mathbf{W}_z$ to $\mathbf{W}_{y+1}$, we can show that Condition α implies Condition β by showing that $\mathrm{A}_{r+1} \cup \mathrm{D}$ is at least as good as $\mathrm{A}_1 \cup \mathrm{B}_1 \cup \ldots \cup \mathrm{B}_r \cup \mathrm{D}$. This suffices since $\mathrm{A}_1, \ldots, \mathrm{A}_{r+1}$, $\mathrm{B}_1, \ldots, \mathrm{B}_r$, and D are arbitrary populations satisfying (6)-(8).

From (3)-(7) and Condition α, it follows that

(9) $\mathrm{A}_{i+1} \cup \mathrm{E}$ is at least as good as $\mathrm{A}_i \cup \mathrm{B}_i \cup \mathrm{E}$ for all i, $1 \leq i \leq r$ and any

E$\subset\mathbf{R}(z, y+1)$,

and from (7) and (8) that

(10) $\mathrm{B}_{i+1}\cup\ldots\cup\mathrm{B}_{r+1}\cup\mathrm{D}\subset\mathbf{R}(z, y+1)$ for all i, $1 \leq i \leq r$.

Consequently, letting $\mathrm{E}=\mathrm{B}_{i+1}\cup\ldots\cup\mathrm{B}_{r+1}\cup\mathrm{D}$, (9) and (10) imply that

(11) $\mathrm{A}_{i+1}\cup[\mathrm{B}_{i+1}\cup\ldots\cup\mathrm{B}_{r+1}\cup\mathrm{D}]$ is at least as good as $\mathrm{A}_{i}\cup\mathrm{B}_{i}\cup[\mathrm{B}_{i+1}\cup\ldots\cup\mathrm{B}_{r+1}\cup\mathrm{D}]$ for all i, $1 \leq i \leq r$ (see Diagram 3).

Transitivity and (11) yield

(12) $\mathrm{A}_{r+1}\cup\mathrm{B}_{r+1}\cup\mathrm{D}$ is at least as good as $\mathrm{A}_{1}\cup\mathrm{B}_{1}\cup\ldots\cup\mathrm{B}_{r+1}\cup\mathrm{D}$

and since $\mathrm{B}_{r+1} = \emptyset$ (7), line (12) is equivalent to (see Diagram 3)

(13) $\mathrm{A}_{r+1}\cup\mathrm{D}$ is at least as good as $\mathrm{A}_{1}\cup\mathrm{B}_{1}\cup\ldots\cup\mathrm{B}_{r}\cup\mathrm{D}$. □

It follows trivially from Lemmas 1.1.1 and 1.1.2 that Lemma 1.1 is true. □

1.4.2. *Lemma 1.2*

Lemma 1.2: The General Non-Extreme Priority Condition implies Condition δ.

Condition δ: For any $\mathbf{W}_z$, $z < 0$, and any number of lives $m > 0$, there is a positive welfare level $\mathbf{W}_u$, and a positive welfare range $\mathbf{R}(1,\ y)$, $u > y$, and a number of lives $n > 0$ such that *if* A$\subset\mathbf{W}_x$, $x \geq u$, B$\subset\mathbf{R}(1, y)$, N(A)=N(B)=n, C$\subset\mathbf{W}_z$, D$\subset\mathbf{W}_3$, N(C)=N(D)=m, *then,* for any E, A∪C∪E is at least as good as B∪D∪E, other things being equal.

We shall prove Lemma 1.2 by first proving

Lemma 1.2.1: The General Non-Extreme Priority Condition implies *Condition* χ.

Condition χ: For any $\mathbf{W}_z$, $z < 0$, there is a positive welfare level $\mathbf{W}_u$, and a positive welfare range $\mathbf{R}(1, y)$, $u > y$, and a number of lives $n > 0$ such that *if* A$\subset\mathbf{W}_x$, $x \geq u$, B$\subset\mathbf{R}(1, y)$, N(A)=N(B)=n, C$\subset\mathbf{W}_z$, D$\subset\mathbf{W}_3$,

N(C)=N(D)=1, *then*, for any E, A∪C∪E is at least as good as B∪D∪E, other things being equal.

Proof: Let

(1) $\mathbf{W}_z$ be any welfare level such that $z < 0$;
(2) $r = 3 - z$;
(3) $\mathbf{W}_{u_i}$ be a positive welfare level, $\mathbf{R}(1, v_i)$ be a positive welfare range, and n_i a number of lives which satisfy the General Non-Extreme Priority Condition for $\mathbf{W}_{z+(i-1)}$ for all i, $1 \le i \le r$;
(4) $\mathbf{W}_u$ be a welfare level such that u equals the maximal element in $\{\mathrm{u}_i$: $1 \le i \le r\}$;
(5) $\mathbf{W}_x$ be a welfare level such that $x \ge u$;
(6) y be a number such that y equals the minimal element in $\{\mathrm{v}_i$: $1 \le i \le r\}$.

Let $\mathrm{A}_1,\ldots,\mathrm{A}_{r+1}$, $\mathrm{B}_0,\ldots,\mathrm{B}_r$, and $\mathrm{C}_1,\ldots,\mathrm{C}_{r+1}$, be any three sequences of populations satisfying

(7) $\mathrm{A}_i \subset \mathbf{W}_x$, $N(\mathrm{A}_i) = n_i$, for all i, $1 \le i \le r$; $\mathrm{A}_{r+1} = \emptyset$;
(8) $\mathrm{B}_i \subset \mathbf{R}(1, y)$, $N(\mathrm{B}_i) = n_i$, for all i, $1 \le i \le r$; $\mathrm{B}_0 = \emptyset$;
(9) $\mathrm{C}_i \subset \mathbf{W}_{z+(i-1)}$, $N(\mathrm{C}_i)$=1, for all i, $1 \le i \le r$+1.

Finally, let

(10) E be any population.

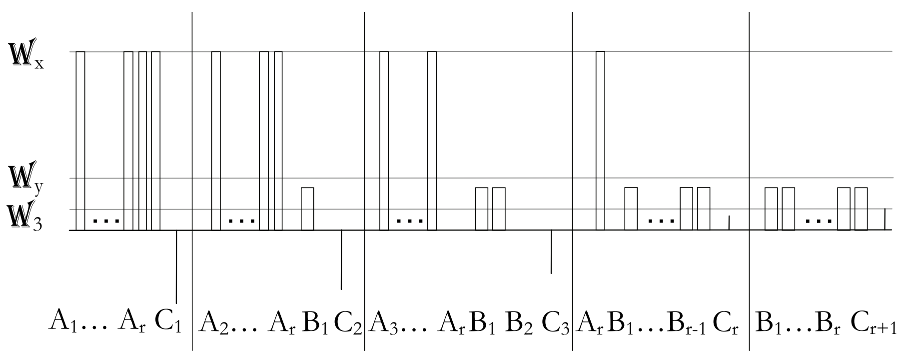


Diagram 4

The above diagram shows a selection of the involved populations in a case where $r \geq 4$. Population E is omitted throughout.

Since $\mathbf{W}_z$ can be any negative welfare level (by (1)), and $\mathbf{W}_x$ can be any welfare level at least as high as $\mathbf{W}_u$ (by (5)), and since it follows from (3) and (6) that $\mathbf{R}(1, y)$ is a welfare range such that $u > y$, we can show that Lemma 1.2.1 is true by showing that $\mathrm{A}_1 \cup \ldots \cup \mathrm{A}_r \cup \mathrm{C}_1 \cup \mathrm{E}$ is at least as good as $\mathrm{B}_1 \cup \ldots \cup \mathrm{B}_r \cup \mathrm{C}_{r+1} \cup \mathrm{E}$. This suffices since $\mathrm{A}_1, \ldots, \mathrm{A}_r$, $\mathrm{B}_1, \ldots, \mathrm{B}_r$, $\mathrm{C}_1, \ldots, \mathrm{C}_{r+1}$, and E are arbitrary populations satisfying (7)-(10).

The General Non-Extreme Priority Condition and (3)-(9) imply that

(11) $\mathrm{A}_i \cup \mathrm{C}_i \cup \mathrm{F}$ is at least as good as $\mathrm{B}_i \cup \mathrm{C}_{i+1} \cup \mathrm{F}$ for all i, $1 \leq i \leq r$ and any population F.

Letting $\mathrm{F} = \mathrm{A}_{i+1} \cup \ldots \cup \mathrm{A}_{r+1} \cup \mathrm{B}_0 \cup \ldots \cup \mathrm{B}_{i-1} \cup \mathrm{E}$, it follows from (11) that

(12) $\mathrm{A}_i \cup \mathrm{C}_i \cup [\mathrm{A}_{i+1} \cup \ldots \cup \mathrm{A}_{r+1} \cup \mathrm{B}_0 \cup \ldots \cup \mathrm{B}_{i-1} \cup \mathrm{E}]$ is at least as good as $\mathrm{B}_i \cup \mathrm{C}_{i+1} \cup [\mathrm{A}_{i+1} \cup \ldots \cup \mathrm{A}_{r+1} \cup \mathrm{B}_0 \cup \ldots \cup \mathrm{B}_{i-1} \cup \mathrm{E}]$ for all i, $1 \leq i \leq r$ (see Diagram 4).

Transitivity and (12) yield

(13) $\mathrm{A}_1 \cup \mathrm{C}_1 \cup \mathrm{A}_2 \cup \ldots \cup \mathrm{A}_{r+1} \cup \mathrm{B}_0 \cup \mathrm{E}$ is at least as good as $\mathrm{B}_r \cup \mathrm{C}_{r+1} \cup \mathrm{A}_{r+1} \cup \mathrm{B}_0 \cup \ldots \cup \mathrm{B}_{r-1} \cup \mathrm{E}$,

and since $\mathrm{A}_{r+1} = \mathrm{B}_0 = \emptyset$ by (7)-(8), line (13) is equivalent to (see Diagram 4)

(14) $\mathrm{A}_1 \cup \ldots \cup \mathrm{A}_r \cup \mathrm{C}_1 \cup \mathrm{E}$ is at least as good as $\mathrm{B}_1 \cup \ldots \cup \mathrm{B}_r \cup \mathrm{C}_{r+1} \cup \mathrm{E}$. $\square$

To show that Lemma 1.2 is true, we now need to prove

Lemma 1.2.2: Condition χ implies Condition δ.

Proof: Let

(1) $\mathbf{W}_z$ be any welfare level such that $z < 0$;
(2) m be any number such that $m > 0$;
(3) $\mathbf{W}_u$ be a positive welfare level, $\mathbf{R}(1, y)$ be a positive welfare range, and n a number of lives which satisfy Condition χ for $\mathbf{W}_z$;
(4) $\mathbf{W}_x$ be a welfare level such that $x \geq u$.

Let $\mathrm{A}_1, \ldots, \mathrm{A}_{m+1}$, $\mathrm{B}_0, \ldots, \mathrm{B}_m$, $\mathrm{C}_1, \ldots, \mathrm{C}_{m+1}$, and $\mathrm{D}_0, \ldots, \mathrm{C}_m$, be any four sequences of populations satisfying

(5) $\mathrm{A}_i \subset \mathbf{W}_x$, $N(\mathrm{A}_i) = n$, for all i, $1 \leq i \leq m$; $\mathrm{A}_{m+1} = \emptyset$;
(6) $\mathrm{B}_i \subset \mathbf{R}(1, y)$, $N(\mathrm{B}_i) = n$, for all i, $1 \leq i \leq m$; $\mathrm{B}_0 = \emptyset$;
(7) $\mathrm{C}_i \subset \mathbf{W}_z$, $N(\mathrm{C}_i)=1$, for all i, $1 \leq i \leq m$; $\mathrm{C}_{m+1} = \emptyset$;
(8) $\mathrm{D}_i \subset \mathbf{W}_3$, $N(\mathrm{D}_i)=1$, for all i, $1 \leq i \leq m$; $\mathrm{D}_0 = \emptyset$.

Finally, let

(9) E be any population.

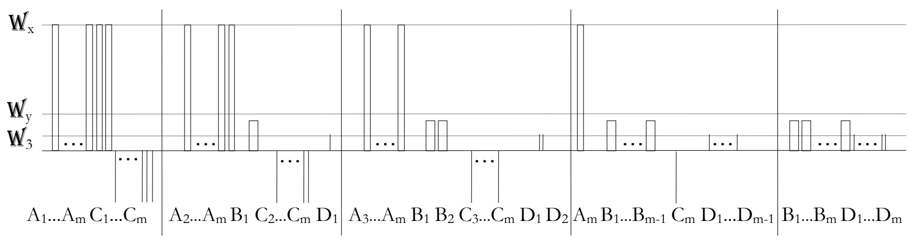


Diagram 5

The above diagram shows a selection of the involved populations in a case where $m \geq 4$. As before, population E is omitted throughout.

Since $\mathbf{W}_z$ can be any negative welfare level (by (1)), and $\mathbf{W}_x$ can be any welfare level at least as high as $\mathbf{W}_u$ (by (5)), and m can be any number of lives greater than zero, and $\mathbf{R}(1, y)$ is a welfare range such that $u > y$, and n is a number greater than zero (by (3)), we can show that Lemma 1.2.2 is true by showing that $\mathrm{A}_1 \cup \ldots \cup \mathrm{A}_m \cup \mathrm{C}_1 \cup \ldots \cup \mathrm{C}_m \cup \mathrm{E}$ is at least as good as $\mathrm{B}_1 \cup \ldots \cup \mathrm{B}_m \cup \mathrm{D}_1 \cup \ldots \cup \mathrm{D}_m \cup \mathrm{E}$. This suffices since $\mathrm{A}_1, \ldots, \mathrm{A}_m$, $\mathrm{B}_1, \ldots, \mathrm{B}_m$, $\mathrm{C}_1, \ldots, \mathrm{C}_m$, $\mathrm{D}_1, \ldots, \mathrm{D}_m$, and E are arbitrary populations satisfying (5)-(9).

It follows from (3)-(8) and Condition χ that

(10) $\mathrm{A}_i \cup \mathrm{C}_i \cup \mathrm{F}$ is at least as good as $\mathrm{B}_i \cup \mathrm{D}_i \cup \mathrm{F}$ for all i, $1 \leq i \leq m$, and any population F

which, for $\mathrm{F}=\mathrm{A}_{i+1} \cup \ldots \cup \mathrm{A}_{m+1} \cup \mathrm{C}_{i+1} \ldots \cup \mathrm{C}_{m+1} \cup \mathrm{B}_0 \cup \ldots \cup \mathrm{B}_{i-1} \cup \mathrm{D}_0 \cup \ldots \cup \mathrm{D}_{i-1} \cup \mathrm{E}$,
in turn implies

(11) $\mathrm{A}_i \cup \mathrm{C}_i \cup [\mathrm{A}_{i+1} \cup \ldots \cup \mathrm{A}_{m+1} \cup \mathrm{C}_{i+1} \ldots \cup \mathrm{C}_{m+1} \cup \mathrm{B}_0 \cup \ldots \cup \mathrm{B}_{i-1} \cup \mathrm{D}_0 \cup \ldots \cup \mathrm{D}_{i-1} \cup \mathrm{E}]$ is at least as good as $\mathrm{B}_i \cup \mathrm{D}_i \cup [\mathrm{A}_{i+1} \cup \ldots \cup \mathrm{A}_{m+1} \cup \mathrm{C}_{i+1} \cup \ldots \cup \mathrm{C}_{m+1} \cup \mathrm{B}_0 \cup \ldots \cup \mathrm{B}_{i-1} \cup \mathrm{D}_0 \cup \ldots \cup \mathrm{D}_{i-1} \cup \mathrm{E}]$ for all i, $1 \leq i \leq m$

(see Diagram 5).

Transitivity and (11) yield

(12) $A_1 \cup C_1 \cup A_2 \cup \ldots \cup A_{m+1} \cup C_2 \ldots \cup C_{m+1} \cup B_0 \cup D_0 \cup E$ is at least as good as $B_m \cup D_m \cup A_{m+1} \cup C_{m+1} \cup B_0 \cup \ldots \cup B_{m-1} \cup D_0 \ldots \cup D_{m-1} \cup E$

and since $A_{m+1}=B_0=C_{m+1}=D_0 = \emptyset$ (by (5)-(8)), line (12) is equivalent to (see Diagram 5)

(13) $A_1 \cup \ldots \cup A_m \cup C_1 \ldots \cup C_m \cup E$ is at least as good as $B_1 \cup \ldots \cup B_m \cup D_1 \ldots \cup D_m \cup E$. □

It follows trivially from Lemmas 1.2.1 and 1.2.2 that Lemma 1.2 is true. □

1.4.3. *Lemma 1.3*

Lemma 1.3: The Weak Quality Addition Condition and Condition δ imply the Restricted Quality Addition Condition.

The Restricted Quality Addition Condition (exact formulation): For any population X, there is a positive welfare level $\mathbf{W}_x$ and a positive welfare range $\mathbf{R}(1,\ y)$, $x\ > y$, and a population size n and m such that *if* $A \subset \mathbf{W}_z$, $z \geq x$, $N(A)=n$, $B \subset \mathbf{R}(1, y)$, $N(B)=p$, $p \geq m$, *then* $A \cup X$ is at least as good as $B \cup X$, other things being equal.

Proof. Let

(1) X be any population;
(2) $\mathbf{R}(w, t)$ and $\mathbf{R}(1, v)$,$w > v$, be two welfare ranges, $\mathbf{W}_z$ a negative welfare level, and p and m two population sizes, which satisfy the Weak Quality Addition Condition for X;
(3) $\mathbf{W}_u$ be a positive welfare level, $\mathbf{R}(1, y)$ a welfare range, and n a number of lives, which satisfy Condition δ for $\mathbf{W}_z$ and m;
(4) Let $\mathbf{W}_x$ be a welfare level such that x=max(w, u);[9]
(5) $A_1 \subset \mathbf{W}_x$, $N(A_1) = n$;
(6) $A_2 \subset \mathbf{W}_x$, $N(A_2) = p$;
(7) $C \subset \mathbf{W}_z$, $N(C)=m$;
(8) $B_1 \subset \mathbf{R}(1,\ \mathrm{v}) \cap \mathbf{R}(1, y)$, $N(B_1) = n$;

[9]We define the function max(x, y) in the ordinary way: max$(x, y) = x$ if $x \geq y$, otherwise max$(x, y) = y$.

(9) $\mathrm{B}_3 \subset \mathbf{W}_3$, $N(\mathrm{B}_3) = m$;
(10) $q \geq 0$ be any population size;
(11) $\mathrm{B}_2 \subset \mathbf{R}(1, v) \cap \mathbf{R}(1, y)$, $N(\mathrm{B}_2) = q$.

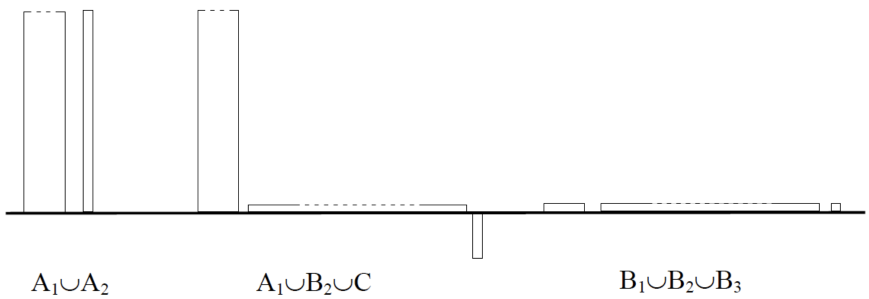


Diagram 6

Population X is omitted throughout in the above diagram.

We can conclude from (8)-(11) that $\mathrm{B}_1 \cup \mathrm{B}_2 \cup \mathrm{B}_3$ can be of any size greater than or equal to $p + m$ since B_2 can be of any size. Moreover, since $\mathrm{A}_1 \cup \mathrm{A}_2 \subset \mathbf{W}_x$, $x > v$ (by (2), (4)-(6)) and $\mathrm{B}_1 \cup \mathrm{B}_2 \cup \mathrm{B}_3 \subset \mathbf{R}(1, v) \cap \mathbf{R}(1, y)$ (by (8), (9), (11)), we can show that Lemma 1.3 is true by showing that $\mathrm{A}_1 \cup \mathrm{A}_2 \cup \mathrm{X}$ is at least as good as $\mathrm{B}_1 \cup \mathrm{B}_2 \cup \mathrm{B}_3 \cup \mathrm{X}$. This suffices since X can be any population (by (1)).

It follows from (2), (4), (6), (11), and the Weak Quality Addition Condition that

(12) $\mathrm{A}_1 \cup \mathrm{A}_2 \cup \mathrm{X}$ is at least as good as $\mathrm{A}_1 \cup \mathrm{B}_2 \cup \mathrm{C} \cup \mathrm{X}$ (see Diagram 6).

It follows from (3)-(5), (7)-(9), and Condition δ that

(13) $\mathrm{A}_1 \cup \mathrm{B}_2 \cup \mathrm{C} \cup \mathrm{X}$ is at least as good as $\mathrm{B}_1 \cup \mathrm{B}_2 \cup \mathrm{B}_3 \cup \mathrm{X}$ (see Diagram 6).

By transitivity, it follows from (12) and (13) that

(14) $\mathrm{A}_1 \cup \mathrm{A}_2 \cup \mathrm{X}$ is at least as good as $\mathrm{B}_1 \cup \mathrm{B}_2 \cup \mathrm{B}_3 \cup \mathrm{X}$. □

1.4.4. *Lemma 1.4*

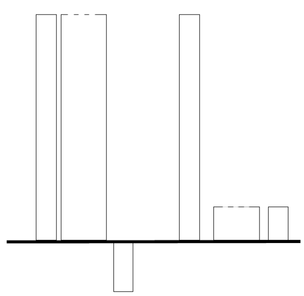
We shall show that the theorem is true by proving

Lemma 1.4: There is no population axiology which satisfies Condition β

and δ, the Egalitarian Dominance, the Restricted Quality Addition, and the Weak Non-Sadism Condition.

***Proof*.** We show that the contrary assumption leads to a contradiction. Let

(1) $\mathbf{W}_z$ be a negative welfare level and m a population size which satisfy the Weak Non-Sadism Condition;
(2) $\mathbf{W}_u$ be a positive welfare level, $\mathbf{R}(1, y)$ a welfare range, and n a number of lives, which satisfy Condition δ for $\mathbf{W}_z$ and m;
(3) $\mathrm{B}_1 \subset \mathbf{W}_3$, $\mathrm{B}_2 \subset \mathbf{W}_3$, $N(\mathrm{B}_1) = n$, $N(\mathrm{B}_2) = m$;
(4) $\mathbf{W}_w$ be a welfare level, and $\mathbf{R}(1, v)$, $w > v$, be a welfare range, and p and k two population sizes, which satisfy the Restricted Quality Addition Condition for $\mathrm{B}_1 \cup \mathrm{B}_2$;
(5) Let $\mathbf{W}_x$ be a welfare level such that $x = \max(w, u)$;
(6) $\mathrm{A} \subset \mathbf{W}_x$, $N(\mathrm{A}) = p$;
(7) $\mathrm{H} \subset \mathbf{W}_x$, $N(\mathrm{H}) = n$;
(8) $\mathrm{E} \subset \mathbf{W}_z$, $N(\mathrm{E}) = m$.



A∪H∪E A∪B$_1$∪B$_2$

Diagram 7

It follows from the definition of a welfare range that $\mathbf{W}_3 \subset \mathbf{R}(1, y)$. Accordingly, from (3) we know that $\mathrm{B}_1 \subset \mathbf{R}(1, y)$. Consequently, from (2), (3), (7), (8), and Condition δ we get that

(9) A∪H∪E is at least as good as A∪B$_1$∪B$_2$ (see Diagram 7).

Let

(10) $r > n + p$ be a number of lives which satisfies Condition β for the three welfare levels $\mathbf{W}_x$, $\mathbf{W}_2$, and $\mathbf{W}_1$ and for $n + p$ lives at $\mathbf{W}_x$;
(11) q be any number of lives such that $q \geq m + k$ and $q \geq r$;
(12) G$\subset\mathbf{W}_2$, N(G)=$n + p + r$;
(13) I$\subset\mathbf{W}_1$, N(I)=$q - r$;
(14) F$\subset\mathbf{W}_1$, N(F)=r.

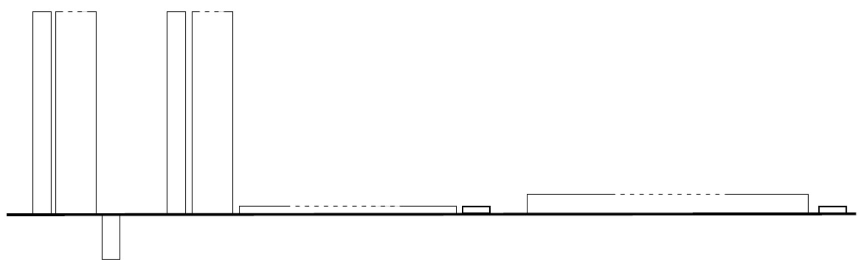


Diagram 8

Since A∪H$\subset\mathbf{W}_x$, and N(A∪H)=$n + p$ (by (6) and (7)), and I$\subset\mathbf{R}$(1, 3) (by the definition of a welfare range), it follows from (10)-(14) and Condition β that

(15) G∪I is at least as good as A∪H∪F∪I (see Diagram 8).

Since the F- and the I-lives have positive welfare (by (13) and (14)), it follows from (1), (8) and the Weak Non-Sadism Condition that

(16) A∪H∪F∪I is at least as good as A∪H∪E (see Diagram 8).

By transitivity, it follows from (15) and (16) that

(17) G∪I is at least as good as A∪H∪E.

Let

(18) C$\subset\mathbf{W}_3$, N(C)=$p + q - m$.

Since $\mathbf{W}_3\subset\mathbf{R}(1, v)$, we can conclude that C$\subset\mathbf{R}(1, v)$ and since $q \geq m + k$

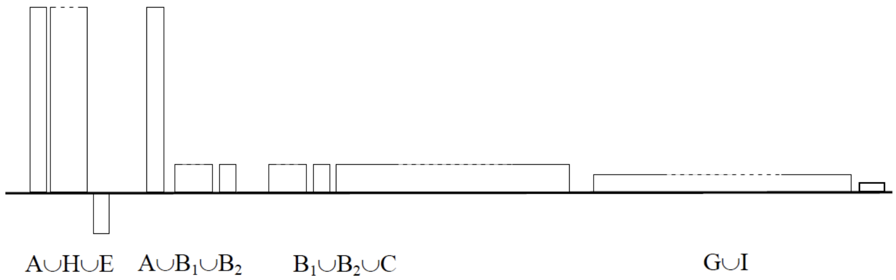


Diagram 9

(by (11)) and N(C)=$p+q-m$ (by (18)), we know that $N(\mathrm{C}) \geq k$. Moreover, since $x \geq w$ (by (5)), and A$\subset\mathbf{W}_x$ (by (6)), it follows from (4) and the Restricted Quality Addition Condition that

(19) A∪B$_1$∪B$_2$ is at least as good as B$_1$∪B$_2$∪C (see Diagram 9).

Since B$_1$∪B$_2$∪C$\subset\mathbf{W}_3$ (by (3) and (18)) and G∪I$\subset\mathbf{W}_1\cup\mathbf{W}_2$, (by (12) and (13)) and N(B$_1$∪B$_2$∪C)=N(G∪I), the Egalitarian Dominance Condition implies that

(20) G∪I is worse than B$_1$∪B$_2$∪C (see Diagram 9).

By transitivity, it follows from (19) and (20) that

(21) G∪I is worse than A∪B$_1$∪B$_2$

and from (9) and (21) that

(22) G∪I is worse than A∪H∪E

which contradicts (17). □

It follows trivially from Lemmas 1.1-1.4 that the impossibility theorem is true. □

1.5. Discussion

The above theorem shows that our considered moral beliefs are mutually inconsistent, that is, necessarily at least one of our considered moral beliefs is false. Since consistency is, arguably, a necessary condition for moral justification, we would thus seem to be forced to conclude that there is no moral theory which can be justified. In other words, the cases in population ethics involving future generations of different sizes constitute a serious challenge to the existence of a satisfactory moral theory.

The theorem presupposes that the relation "is at least as good as" is transitive. Some theorists find this a matter of logic, claiming that it is part of the meaning of "better than" and "equally as good as" (Broome, 1991, p. 11). Although we are inclined to agree, one might think otherwise, and argue that the impossibility theorem actually demonstrates that these relations are non-transitive.[10] What is attractive with this move is that given non-transitivity of "at least as good as" and "better than", we can stick to our axiological evaluations without any contradiction.[11] However, as I have shown elsewhere, the axiological population theorems, including the above one, can be reconstructed on the normative level, in terms of what one ought to choose, without any appeal to transitivity (see Arrhenius 2004, 2011). Instead of a non-transitive ordering of populations, one gets a situation in which all of the available actions are forbidden, i.e., a moral dilemma. Hence, it does not look like we can exorcise the paradoxes of population ethics by giving up some formal condition like the transitivity of "better than".

In our discussion we have assumed that welfare is at least sometimes interpersonally comparable. Without this assumption, claims such as "Iwao is better off than Ben" would not be meaningful. In other words, conditions such as the Egalitarian Dominance and the Non-Elitism Condition, in

[10] Among others, Larry Temkin and Stuart Rachels have suggested something like this, see Rachels (1998, 2001) and Temkin (1987, 1996). See Broome (2004), Section 4.1, for a thorough discussion of various arguments against the transitivity of betterness, including Temkin's and Rachel's arguments.

[11] However, as many seem to fear (including Temkin), non-transitivity of "better than" might spell the end for any axiology-based morality and practical reason in general. Temkin suggests that arguments to the effect that "better than" is non-transitive "are [perhaps] best interpreted as a frontal assault on the intelligibility of consequentialist reasoning about morality and rationality. Such reasoning may need to be severely limited, if not jettisoned altogether." (Temkin 1987, p. 186, fn. 49). Elsewhere, he considers that non-transitivity "...opens the possibility that there would be no rational basis for choosing between virtually any alternatives" (Temkin 1996, p. 209). This needs to be shown, however. I argue the contrary in Arrhenius (2011), Chapter 12.

their normative or axiological guise, would not make sense. The adequacy conditions and the theorems are quite undemanding, however, in regard to measurement of welfare. It does not matter whether welfare is measurable on an ordinal, interval or ratio scale, for example. The conditions and theorems only presuppose that lives are quasi-ordered by the relation "has at least as high welfare as".

It is interesting to compare the information demands of the present theorems with that of Arrow's famous impossibility theorem (see Arrow, 1963). Without interpersonal comparability of welfare, one gets Arrowian impossibility results already in a fixed population size setting.[12] Not surprisingly then, the standard remedy for such impossibility results is to introduce some kind of interpersonal comparability of welfare.[13] But with interpersonal comparability of welfare, and some minimal demands on the orderings of lives, we come up against the impossibility theorem presented here. Moreover, and more worryingly, this result holds even if we have complete interpersonal comparability of welfare on a ratio scale, that is, access to all the possible information about people's welfare for which we could wish.

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[12]See Sen (1970), pp. 123-125, 128-130, and Roemer (1996), pp. 26-36.
[13]Roemer (1996), p. 36, among many others, suggests this.

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